Mon. Not. R. Astron. Soc. **000**, 1–10 (0000) Printed 1 February 2008 (MN LATEX style file v1.4)

Dynamical friction for compound bodies

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Accepted ..... Received .....; in original form 1997 May 14

ABSTRACT

In the framework of the fluctuation-dissipation approach to dynamical friction, we derive an expression giving the orbital energy exchange experienced by a compound body as it moves interacting with a non homogeneous discrete background. The body is assumed to be composed of particles endowed with a velocity spectrum and with a non homogeneous spatial distribution. The Chandrasekhar formula is recovered in the limit of a point-like satellite with zero velocity dispersion and infinite temperature moving through an homogeneous infinite medium. In this same limit, but dropping the zero satellite velocity dispersion (σ_S) condition, the orbital energy loss is found to be smaller than in the $\sigma_S = 0$ case by a factor of up to an order of magnitude in some situations.

Key words: methods: analytical – celestial mechanics, stellar dynamics, galaxy dynamics.

1 INTRODUCTION

A satellite moving through a field of gravitating particles experiences a dissipative frictional force known as dynamical friction. It can be understood in terms of the satellite wake, that exerts a drag force on the satellite itself (Kalnajs 1972; Binney & Tremaine 1987; Weinberg 1989; Bekenstein & Zamir 1991). It can also been understood in terms of the underlying basic physics as the friction resulting from the fluctuating gravitating forces acting on the satellite as a consequence of the non-continuos character of the particle system. The fact that fluctuating forces cause dissipation is quite a general scheme in Physics. It is at the basis of physical phenomena such as electric resistance in conductors or viscous friction in liquids (Reif 1965).

Dynamical friction has important consequences in the evolution of astronomical systems, mainly because it causes a decay of orbiting bodies, so that merger timescales and dissipation rates by dynamical friction are closely related. The merger scenario is at the basis of a great deal of processes in Astronomy. Not only is it the framework for the general galaxy formation picture in hierarchical cosmological models, but also for more particular aspects of the evolution of a number of astronomical systems, such as galactic nuclei, cD galaxies in rich galaxy clusters, compact galaxy groups and so on.

A dynamical friction formula was first obtained, in a kinematical approach, by Chandrasekhar (1943). He has calculated the rate of momentum exchange between the test and field particles as the result of a sum of uncorrelated two-body encounters, obtaining the expression:

$$M\frac{d\boldsymbol{v}}{dt} = -\frac{4\pi G^2 M^2}{v^3}\ln\Lambda\,\rho(<v)\,\boldsymbol{v}, \tag{1}$$

where M and $\boldsymbol{v}$ are the test particle mass and speed, respectively, $\rho(< v)$ is the density of background particles with velocity less than v, G is the gravitation constant and $\Lambda = \frac{p_{max}}{p_{min}}$, with p_{max} and p_{min} the maximum and minimum impact parameters contributing to the drag. Keplerian orbits for both the test and field particles and an infinite and homogeneous background have been assumed in the derivation of eq. (1)

Chandrasekhar's formula is widely employed to quantify dynamical friction in a variety of situations, even if in most astronomical problems the background is neither infinite nor homogeneous. This formula is known to give the correct order of magnitude, but it suffers from several drawbacks, that arise from the very physical assumptions made in its derivation. Furthermore, it cannot describe some situations, as for example the drag experienced by a satellite placed outside the edge of a finite gravitating system. In fact, the satellite would be decelerated on physical grounds, because it causes a perturbation to the system. However, according to eq. (1), the drag would vanish. For this reason, other works on dynamical friction followed Chandrasekhar's pioneering study, either from a numerical (Lin & Tremaine 1983; White 1983; Bontekoe & van Albada 1987; Zaritsky & White 1988) or an analytical point of view. Analytical descriptions have

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the advantage that they help understanding the underlying physics. Several methods have been developed: Fokker-Planck equation based (Rosenblunth, Mc Donald & Judd 1957; Binney & Tremaine 1987) polarization cloud approach (Marochnik 1968; Kalnajs 1972; Binney & Tremaine 1987; Weinberg 1989; Bekenstein & Zamir 1991), resonant particle interactions (Lynden-Bell & Kalnajs 1972; Tremaine & Weinberg 1984; Palmer & Papaloizou 1985; Weinberg 1986).

Berkenstein and Maoz (1992, hereafter BM92) and Maoz (1993, hereafter M93) introduced a fluctuation-dissipation approach to dynamical friction. The fluctuation-dissipation theorem (Kubo 1959) relates the friction coefficient to the time integral of the correlation function for the fluctuations causing the friction. They showed that dynamical friction fits into this general scheme, which provides a powerful technique to study it. This approach is in fact a return to Chandrasekhar's original attempt to give a statistical description of dynamical friction. Other stochastic approaches to dynamical friction are those of Cohen (1975) and Kandrup (1980). M93 derived a formula for the drag experienced by an object which travels in an arbitrary mass density field, assumed to be stationary and formed by particles much lighter than the object.

A common feature of all the previous approaches is that they treat the satellite as being rigid and without structure in the velocity space. This approximation can be good enough in a number of astronomical situations, but in others, these two ingredients could play a crucial role. As a first example of such a situation, let us consider the dynamical evolution of compact groups of galaxies (e.g. Mamon 1993). In this case, the velocity dispersion of individual galaxies is comparable to the velocity dispersion of the common halo that hosts them. As a second example, we recall that in the problem of the interaction of two comparable mass galaxies, the energy exchange due to dynamical friction cannot be calculated in the previous frameworks, because their presumably comparable velocity dispersions need to be taken into account.

In this paper we present a fluctuation-dissipation study of the orbital changes experienced by a non-rigid satellite, composed of gravitating particles with a finite velocity dispersion, as it interacts with a general background. An extension of BM92 and M93 techniques has allowed us to calculate the rate of energy exchange between them as a result of fluctuations in the gravitational forces of both the background and the satellite.

The paper is organized as follows: in Sect. 2, the physical formulation of the method and the general expressions giving the energy exchange rate are presented. In Sect. 3, we calculate the instantaneous energy variations for general backgrounds at rest and with a Maxwellian velocity distribution. Some particular limits are dealt with in Sect. 4. Finally, in Sect. 5, we summarize and discuss our results. Two Appendices follow, where the results of the calculation of the correlation matrix and of an integral needed are given.

2 PHYSICAL FORMULATION

We will study the energy exchange between two self-gravitating equilibrium systems called the background (B) and the satellite (S). The background consists of N_B equal mass particles, each of them with a mass m_B. Its total mass and typical size are $M_B = m_B N_B$ and R_B, respectively. These particles exhibit a velocity spectrum with dispersion σ_B and zero mean at $t = 0$ (which is equivalent to considering the origin of the reference system placed at the halo center of mass at $t = 0$). The satellite in turn is composed of N_S equal mass particles of mass m_S, total mass M_S, typical size R_S, and a velocity distribution with dispersion σ_S and mean equal to the center of mass velocity of the satellite, $\boldsymbol{v}_{CMS}$.

As we are mainly interested in the effects that a non vanishing σ_S would have on the evolution of the whole system, we can consider that $\sigma_S \sim \sigma_B \equiv \sigma$. The virial theorem tells us that $\sigma \sim GNmR^{-1}$, so that $\sigma_S \sim \sigma_B$ implies that

$$N_S m_S R_S^{-1} \sim N_B m_B R_B^{-1}. \tag{2}$$

As both the satellite and the background are assumed to have a finite size, we can safely consider that

$$N_\alpha^{1/n} \gg \left(\frac{R_\beta}{R_\gamma}\right)^m \tag{3}$$

for whatever $\alpha, \beta, \gamma = B$ or S, and n, m of order unity. If $m_\alpha \gg m_{\tilde{\alpha}}$ ($\tilde{\alpha}$ is the particle class contrary to α), as could be the case, for example, when the α and $\tilde{\alpha}$ systems are composed of stars and dark matter particles, respectively, then eqs. (2) and (3) imply the inequality $N_\alpha \ll N_{\tilde{\alpha}}$. But, in any case, both N_α and $N_{\tilde{\alpha}}$ will be assumed to be very large.

2.1 The fluctuating forces acting on a particle

The fluctuating force, $\boldsymbol{F}_{i_S}(\boldsymbol{r}_{i_S}, t)$, acting at time t on a satellite particle i_S placed at position $\boldsymbol{r}_{i_S}$ and caused by its interactions with both background and satellite particles, can be written as:

$$\boldsymbol{F}_{i_S}(\boldsymbol{r}_{i_S}, t) = \boldsymbol{F}^B_{i_S}(\boldsymbol{r}_{i_S}, t) + \boldsymbol{F}^S_{i_S}(\boldsymbol{r}_{i_S}, t) \tag{4}$$

where

$$\begin{aligned} \boldsymbol{F}^B_{i_S}(\boldsymbol{r}_{i_S}, t) &= -G m_S m_B \boldsymbol{\nabla} \left(\sum_{j_B} \frac{1}{\mid \boldsymbol{r}_{i_S} - \boldsymbol{r}_{j_B} \mid} \right. \\ &\quad - \left. \int \frac{d\boldsymbol{r} n_B(\boldsymbol{r})}{\mid \boldsymbol{r}_{i_S} - \boldsymbol{r} \mid} \right) \end{aligned} \tag{5}$$

and

$$\begin{aligned} \boldsymbol{F}^S_{i_S}(\boldsymbol{r}_{i_S}, t) &= -G m_S^2 \boldsymbol{\nabla} \left(\sum_{j_S \neq i_S} \frac{1}{\mid \boldsymbol{r}_{i_S} - \boldsymbol{r}_{j_S} \mid} \right. \\ &\quad - \left. \frac{N_S - 1}{N_S} \int \frac{d\boldsymbol{r} n_S(\boldsymbol{r})}{\mid \boldsymbol{r}_{i_S} - \boldsymbol{r} \mid} \right) \end{aligned} \tag{6}$$

are the fluctuating forces acting on i_S caused by the background and satellite, respectively. Each of these forces results from subtracting to the total many-body or discrete force a smooth part derived from the mean field potentials, $\Phi_B(\boldsymbol{r})$ and $\Phi_S(\boldsymbol{r})$, due to the smooth densities, $n_B(\boldsymbol{r})$ and $n_S(\boldsymbol{r})$, respectively. These satisfy the relations:

$$n_\alpha(\boldsymbol{r}) = N_\alpha \int d\boldsymbol{u} f_0^\alpha(\boldsymbol{r}, \boldsymbol{u}), \tag{7}$$

$$\nabla^2 \Phi_\alpha(\boldsymbol{r}) = 4\pi G m_\alpha n_\alpha(\boldsymbol{r}) \tag{8}$$

and

$$\int d\boldsymbol{r} n_\alpha(\boldsymbol{r}) = N_\alpha, \tag{9}$$

where $\alpha = B, S$, $f_0^\alpha(\boldsymbol{r},\boldsymbol{u})$ is the one-particle distribution function for background and satellite particles in the unperturbed state, and the $\frac{N_S-1}{N_S}$ factor takes into account that the i_S particle does not interact with itself at time t. An exchange of the B and S labels in eq. (5) gives the expression for the fluctuating force on the background particle i_B caused by the satellite. Changing the S labels into B in eq. (6) we get the force on i_B caused by the background fluctuations. In a compact formulation, we can write the fluctuating force on a generic class α particle, i_α, due to class β particles as:

$$\begin{aligned} \boldsymbol{F}^\beta_{i_\alpha}(\boldsymbol{r}_{i_\alpha},t) &= -G m_\alpha m_\beta \boldsymbol{\nabla} \left(\sum_{j_\beta \neq i_\alpha} \frac{1}{\mid \boldsymbol{r}_{i_\alpha} - \boldsymbol{r}_{j_\beta} \mid} \right. \\ &- \left. \frac{N'_\beta}{N_\beta} \int \frac{d\boldsymbol{r} n_\beta(\boldsymbol{r})}{\mid \boldsymbol{r}_{i_\alpha} - \boldsymbol{r} \mid} \right) \end{aligned} \tag{10}$$

where now $N'_\beta = N_\beta - \delta_{\alpha\beta}$ takes into account the possibility that $\alpha = \beta$.

The equation of motion of either a background or a satellite particle at time t can be written in a compact notation as:

$$\begin{aligned} \boldsymbol{v}_{i_\alpha}(t) &= \boldsymbol{v}_{i_\alpha,0} + \sum_\beta \left[\frac{1}{m_\alpha} \int_0^t \boldsymbol{F}^\beta_{i_\alpha}(\boldsymbol{r}_{i_\alpha},t')dt' \right. \\ &- \left. \int_0^t \boldsymbol{\nabla}\Phi_\beta(\boldsymbol{r}_{i_\alpha},t')dt' \right], \end{aligned} \tag{11}$$

where $\boldsymbol{v}_{i_\alpha,0}$ is the particle velocity at time $t=0$.

Stochastic forces are weak as compared with the smooth global forces. In fact, the fluctuating force between a class α and a class β particle is of order (Kandrup 1980, Saslaw 1985)

$$\mid \boldsymbol{F}^\beta_{i_\alpha,fluc} \mid = O\left(G m_\alpha m_\beta n^{2/3}\right), \tag{12}$$

where $n \sim N_\alpha R_\alpha^{-3} + (1-\delta_{\alpha\beta}) N_\beta R_\beta^{-3}$ is the average particle number density for the whole system. The smooth density of class δ particles causes a gravitating force on a class γ particle that is of order:

$$\mid \boldsymbol{F}^\delta_{j_\gamma,smooth} \mid = O\left(G m_\gamma m_\delta N_\delta R_\delta^{-2}\right). \tag{13}$$

Eqs. (12) and (13) give, after some algebra:

$$\begin{aligned} \frac{\mid \boldsymbol{F}^\beta_{i_\alpha,fluc} \mid}{\mid \boldsymbol{F}^\delta_{j_\gamma,smooth} \mid} &= O\left(\left[\left(\frac{m_\beta}{m_\delta}\right)^{1/2} \left(\frac{R_\delta}{R_\beta}\right)^2 \right.\right. \\ &+ \left.\left. (1-\delta_{\alpha\beta}) \left(\frac{m_\beta}{m_\delta}\right)^{3/2} \right]^{2/3} \frac{m_\alpha}{m_\gamma} N_\delta^{-1/3} \right), \end{aligned} \tag{14}$$

where eq. (2) has been taken into account. Now, if $m_S \sim m_B$, then we will have

$$\frac{\mid \boldsymbol{F}^\beta_{i_\alpha,fluc} \mid}{\mid \boldsymbol{F}^\delta_{j_\gamma,smooth} \mid} = O\left(\left[\left(\frac{R_\delta}{R_\beta}\right)^2 + (1-\delta_{\alpha\beta}) \right]^{2/3} N_\delta^{-1/3} \right), \tag{15}$$

this is indeed a very small quantity. If, on the contrary, background and satellite particles have very different masses, eq. (14) must be examined more carefully. The most unfavorable case is when $m_\delta \ll m_{\tilde{\delta}}, \gamma = \delta$ and $\alpha = \beta = \tilde{\delta}$. In this case, eq. (14) gives:

$$\frac{\mid \boldsymbol{F}^\beta_{i_\alpha,fluc} \mid}{\mid \boldsymbol{F}^\delta_{j_\gamma,smooth} \mid} = O\left(\left[\frac{m_\beta}{m_\delta}\right]^{4/3} \left[\frac{R_\delta}{R_\beta}\right]^{4/3} N_\delta^{-1/3} \right), \tag{16}$$

and it would be sufficient to ask that

$$N_\delta^{1/3} \gg \left(\frac{m_{\tilde{\delta}}}{m_\delta}\right)^{4/3} \tag{17}$$

to ensure that the fluctuating forces are much smaller than the smooth ones.

Next we compare the timescale for fluctuation, $\tau_{\alpha\beta}$, and the timescale $T_{\gamma\delta}$ over which the velocity of class γ particles changes appreciably as a result of the smooth forces produced by class δ particles. The first is of order of the time required for the nearest neighbor to travel an inter-particle distance, $\tau_{\alpha\beta} = \frac{1}{\sigma n^{1/3}}$, where σ is σ_α or $\sigma_{\tilde{\alpha}}$ and $n = n_\alpha + (1-\delta_{\alpha\beta}) n_\beta$. As it may happen that n_α and n_β are very different, we take the maximum $\tau_{\alpha\beta}$ timescale, corresponding to $n_{min} = N_{min} R_{min}^{-3}$. The second timescale is set by $T_{\gamma\delta} = \frac{|\boldsymbol{v}_\gamma|}{|\boldsymbol{\nabla}\Phi_\delta|} \sim \frac{|\boldsymbol{v}_\gamma| R_\delta}{\sigma^2}$, so that we can write

$$\frac{\tau^{max}_{\alpha\beta}}{T_{\gamma\delta}} \sim O\left(N_{min}^{-1/3} \frac{R_{min}}{R_\delta}\right) \ll 1, \tag{18}$$

where we have used the inequality (3).

The rate of energy exchange between subsystems B and S will be obtained as an integral on time which involves the correlation matrix (see below and BM92 and M93). The correlation matrix is known to fall with time faster than t^{-1} (Cohen 1975; Kandrup 1980; M93), so that it can be taken to vanish for times much larger than the fluctuation timescale, and in particular for times of the order of the macroscopic timescale, $T_{\gamma\delta}$. As a consequence, in this work we will consider time intervals after $t=0$, δt, which are large as compared with the fluctuation timescale, $\tau^{max}_{\alpha\beta}$, but much shorter than the macroscopic timescale (see Reif 1965 and M93 for a detailed discussion)

$$\tau^{max}_{\alpha\beta} \ll \delta t \ll min\left[\frac{\mid \boldsymbol{v}_\alpha \mid}{\mid \frac{d}{dt}\boldsymbol{v}_\alpha \mid}, \frac{\mid \boldsymbol{v}_\alpha \mid}{\mid \boldsymbol{\nabla}\Phi_\gamma \mid} \right]. \tag{19}$$

2.2 The energy exchange

The energy of a particle i_α is not conserved. Its total instantaneous variation at time t, due to the fluctuating forces caused by class β particles is given by:

$$\left(\frac{dE^\beta_{i_\alpha}}{dt}\right)_t = \boldsymbol{F}^\beta_{i_\alpha}(t) \cdot \boldsymbol{v}_{i_\alpha}(t). \tag{20}$$

Taking into account the expression for $\boldsymbol{v}_{i_\alpha}(t)$ given by eq. (11), we get for times t verifying (19)

$$\left(\frac{dE^\beta_{i_\alpha}}{dt}\right)_t = \boldsymbol{F}^\beta_{i_\alpha}(t) \cdot \left[\boldsymbol{v}_{i_\alpha,0} + \frac{1}{m_\alpha} \sum_\gamma \int_0^t dt' \boldsymbol{F}^\gamma_{i_\alpha}(t') \right], \tag{21}$$

where the potential terms have been neglected because $\mid \boldsymbol{v}_{i_\alpha,0} \mid \gg \mid \boldsymbol{\nabla}\Phi_\gamma \mid t$, as ensured by (19). The total energy change of the i_α particle during the time interval $(0, \delta t)$ is easily calculated by integrating eq. (21). Summing up on i_α we get the total energy variation of class α particles due to the fluctuating forces caused by β particles in this time interval:

$$\Delta E^{\beta}_{\alpha,tot}(\delta t) = \sum_{i_\alpha} \int_0^{\delta t} dt \boldsymbol{F}^{\beta}_{i_\alpha}(t) \cdot \left[\boldsymbol{v}_{i_\alpha,0} + \frac{1}{m_\alpha} \sum_\gamma \int_0^t dt' \boldsymbol{F}^{\gamma}_{i_\alpha}(t') \right]. \quad (22)$$

We will be interested in the average value of this quantity.

2.3 Statistical averaging

The presence and motion of the i_S satellite particle perturbs the background. As a result, while the statistical (ensemble) average of the background fluctuating forces acting on the satellite vanishes in the unperturbed state, $\langle \boldsymbol{F}^B_{i_S}(t)\rangle_0 = 0$, they do not vanish anymore in the real perturbed system, $\langle \boldsymbol{F}^B_{i_S}(t)\rangle \neq 0$. The same is true in general for the stochastic forces caused by class β particles and acting on class α particles: $\langle \boldsymbol{F}^{\beta}_{i_\alpha}(t)\rangle_0 = 0$, but $\langle \boldsymbol{F}^{\beta}_{i_\alpha}(t)\rangle \neq 0$, because the phase density and the number of microstates available to the system has changed due to the perturbation. Following BM92 and M93, we assume that the probability of finding a dynamical variable, Q, with a given value, $P[Q]$, is proportional to the change in the number of microstates available to the whole system. This change is given by the factor:

$$K = \exp\left[\delta S\right], \quad (23)$$

where δS is the entropy change of the system as a result of the distortion. The expected value of a dynamical function, Q, can now be written as:

$$\langle Q \rangle = \frac{\int d\Gamma Q(\Gamma) f_0(\Gamma) K}{\int d\Gamma f_0(\Gamma) K}, \quad (24)$$

where Γ are the variables defining the phase space of the $N_B + N_S$ particles and $f_0(\Gamma)$ is the distribution function for the unperturbed state.

As a result of the fluctuations, a generic particle initially with energy ε_{i_α}, will change to an energy $\varepsilon'_{i_\alpha} = \varepsilon_{i_\alpha} + \delta\varepsilon_{i_\alpha}$, $\alpha = B, S$. Recalling the definition of entropy:

$$S = S_0 - \int f \ln f d\Gamma, \quad (25)$$

with S_0 a constant, this energy change results into a total entropy variation given by:

$$\delta S = \sum_\alpha \left[-\sum_{i_\alpha} \ln f^\alpha(\varepsilon'_{i_\alpha}) + \sum_{i_\alpha} \ln f^\alpha(\varepsilon_{i_\alpha}) \right], \quad (26)$$

where $f^\alpha(\varepsilon_{i_\alpha})$ is the class α one particle distribution function for the unperturbed state corresponding to an energy ε_{i_α}. Most two-body encounters are weak, and then $\delta\varepsilon_{i_\alpha} \ll \varepsilon_{i_\alpha}$. Expanding the r.h.s. of eq.(26) and then the exponential in eq.(23), we obtain that the change in the probability function is given approximately by:

$$K = 1 - \sum_\alpha \sum_{i_\alpha} \frac{1}{f^\alpha} \frac{\partial f^\alpha}{\partial \varepsilon_{i_\alpha}} \delta\varepsilon_{i_\alpha}. \quad (27)$$

For most astronomical applications it is justified to use an isothermal Maxwellian one-particle distribution function,

$$f^\alpha_0(\boldsymbol{r}, \boldsymbol{u}) = \frac{n_\alpha(\boldsymbol{r})}{N_\alpha \left(2\pi\sigma_\alpha^2\right)^{3/2}} \exp\left[-\beta_\alpha \varepsilon_{i_\alpha}\right], \quad (28)$$

where $\boldsymbol{u}$ is the velocity vector, σ_α is the velocity dispersion of class α particles and $\beta_\alpha = \frac{1}{m_\alpha \sigma_\alpha^2}$ is an inverse temperature. Introducing this expression for the distribution function in eq. (27), the factor of change becomes:

$$K = 1 + \sum_\alpha \beta_\alpha \sum_{i_\alpha} \delta\varepsilon_{i_\alpha}. \quad (29)$$

The next step is to find out an expression for the energy variation. The total energy of the system must be conserved, so that we have:

$$\sum_\alpha \sum_{i_\alpha} \delta\varepsilon_{i_\alpha} = 0. \quad (30)$$

Moreover, for one given particle, i_α, its energy variation can be expressed as:

$$\delta\varepsilon_{i_\alpha} = \Delta E^{\alpha}_{i_\alpha} + \Delta E^{\tilde\alpha}_{i_\alpha} - a_{i_\alpha} \Delta E^{\alpha}_{\alpha,tot} - b_{i_\alpha} \Delta E^{\alpha}_{\tilde\alpha,tot}, \quad (31)$$

where a_{i_α} is the fraction of the total class α autointeraction energy, $\Delta E^{\alpha}_{\alpha,tot}$, absorbed by particle i_α, and b_{i_α} is the fraction of the total interaction energy between class α and $\tilde\alpha$ particles, caused by the fluctuating forces of the α subsystem, absorbed by i_α particle. Summing on i_α in eq. (31), we get the total energy change of subsystem α:

$$\sum_{i_\alpha} \delta\varepsilon_{i_\alpha} = \Delta E^{\tilde\alpha}_{\alpha,tot} - \Delta E^{\alpha}_{\tilde\alpha,tot}, \quad (32)$$

compatible with energy conservation (eq. (30)). To write down eq. (32) from (31), it has been taken into account that $\sum_{i_\alpha} a_{i_\alpha} = \sum_{i_\alpha} b_{i_\alpha} = 1$. Energy signs are such that $\Delta E^{\gamma}_{i_\alpha} > 0$ if particle i_α gains energy due to the fluctuating forces caused by class γ particles and conversely. With this convention, if energy flux is from subsystem α to subsystem $\tilde\alpha$, then $\Delta E^{\tilde\alpha}_{\alpha,tot} < 0$ (subsystem α loses energy), and $\Delta E^{\alpha}_{\tilde\alpha,tot} > 0$ (subsystem $\tilde\alpha$ gains energy), so that $\sum_{i_\alpha} \delta\varepsilon_{i_\alpha} < 0$ and $\sum_{i_{\tilde\alpha}} \delta\varepsilon_{i_{\tilde\alpha}} > 0$. Inserting eq. (32) in eq. (29) and recalling eq. (22), we obtain:

$$K = 1 + \sum_\alpha \beta_\alpha \left\{ \sum_{i_\alpha} \int_0^{\delta t} dt \boldsymbol{F}^{\tilde\alpha}_{i_\alpha}(t) \cdot \left[\boldsymbol{v}_{i_\alpha,0} + \frac{1}{m_\alpha} \sum_\gamma \int_0^t dt' \boldsymbol{F}^{\gamma}_{i_\alpha}(t') - \sum_{i_{\tilde\alpha}} \int_0^{\delta t} dt \boldsymbol{F}^{\alpha}_{i_{\tilde\alpha}}(t) \cdot \left[\boldsymbol{v}_{i_{\tilde\alpha},0} + \frac{1}{m_{\tilde\alpha}} \sum_\gamma \int_0^t dt' \boldsymbol{F}^{\gamma}_{i_{\tilde\alpha}}(t') \right] \right] \right\}. \quad (33)$$

Eq. (24) allows us now to write the ensemble average of the instantaneous energy variation at time t of a generic class μ particle due to the stochastic forces caused by class ν particles:

$$\left\langle \left(\frac{dE^{\nu}_{i_\mu}}{dt} \right)_{\delta t} \right\rangle = \int d\Gamma f_0(\Gamma) \Bigg[\boldsymbol{F}^{\nu}_{i_\mu}(\delta t) \cdot \Bigg(\boldsymbol{v}_{i_\mu,0}$$
$$+ \frac{1}{m_\mu} \sum_\rho \int_0^{\delta t} dt \boldsymbol{F}^{\rho}_{i_\mu}(t) \Bigg) \Bigg] \Bigg[1 + \sum_\alpha \beta_\alpha \Bigg\{ \sum_{i_\alpha} \int_0^{\delta t} dt \boldsymbol{F}^{\tilde\alpha}_{i_\alpha}(t)$$
$$\cdot \Bigg(\boldsymbol{v}_{i_\alpha,0} + \frac{1}{m_\alpha} \sum_\gamma \int_0^t dt' \boldsymbol{F}^{\gamma}_{i_\alpha}(t') \Bigg) - \sum_{i_{\tilde\alpha}} \int_0^{\delta t} dt \boldsymbol{F}^{\alpha}_{i_{\tilde\alpha}}(t)$$
$$\cdot \Bigg(\boldsymbol{v}_{i_{\tilde\alpha},0} + \frac{1}{m_{\tilde\alpha}} \sum_\gamma \int_0^t dt' \boldsymbol{F}^{\gamma}_{i_{\tilde\alpha}}(t') \Bigg) \Bigg\} \Bigg] \left[\int d\Gamma f_0(\Gamma) K \right]^{-1} \quad (34)$$

To second order in the fluctuating forces, taking into account that the average of the stochastic forces in the unperturbed state vanishes, and summing on the i_μ subindex to obtain the global effect, we get:

$$\left\langle \left(\frac{dE^{\nu}_{\mu,tot}}{dt} \right)_{\delta t} \right\rangle = \sum_{i_\mu} \frac{1}{m_\mu} \Bigg[\int_{-\delta t}^0 ds Tr \left[C^{\nu\nu}_{i_\mu i_\mu}(s) \right]$$
$$+ (\beta_{\tilde\nu} - \beta_\nu) \sum_{j_{\tilde\nu}} \int_{-\delta t}^0 ds \boldsymbol{v}_{i_\mu,0} C^{\nu\nu}_{i_\mu j_{\tilde\nu}}(s) \boldsymbol{v}_{j_{\tilde\nu},0} \Bigg], \quad (35)$$

where

$$C^{\alpha\beta}_{i_\gamma j_\delta}(s) = \langle \boldsymbol{F}^{\alpha}_{i_\gamma}(0) \otimes \boldsymbol{F}^{\beta}_{j_\delta}(s) \rangle_0 \quad (36)$$

is the correlation matrix (see Appendix A; the symbol $\otimes$ stands for the tensorial product of vectors $\boldsymbol{F}^{\alpha}_{i_\gamma}$ and $\boldsymbol{F}^{\beta}_{j_\delta}$, the average is on the unperturbed states of particles α and β and the invariance of the correlation matrix under time translation has been taken into account). Because in the unperturbed state two different particles are not correlated, the correlation matrix defined in the previous equation vanishes if $\alpha \neq \beta$. This has been taken into account to deduce eq. (35) from eq. (34). Note that because for $s \geq \delta t$ the correlations vanish, the lower limit of the integrals can be extended up to $-\infty$. Eq. (35) is the expression of the global instantaneous energy variation of subsystem μ due to the fluctuating forces of subsystem ν. The integrand in the first term of the r.h.s. of eq. (35) is invariant under time reversal. Extending the integral to positive s, this term is the corresponding power spectrum at zero frequency (Wiener-Khintchine theorem, see Reif 1965 and BM92) and, consequently, it is positive and represents a heating term of class μ particles due to the fluctuating forces caused by ν particles. It is of order $O(\frac{1}{N}) \ll 1$ relative to the second term. Terms of this kind will be neglected in this work. The second term can be either positive or negative, depending on the sign of $(\beta_{\tilde\nu} - \beta_\nu)$. In the next section it will be explicitly calculated for $\mu, \nu = B, S$.

3 THE INSTANTANEOUS ENERGY VARIATION

3.1 Energy exchange between the satellite and the background

Energy flows between the satellite and the background as a result of either the global energy variation of the satellite particles due to the fluctuating forces of the background, or, conversely, the global energy variation of the background particles due to the fluctuating forces of the satellite. These fluxes are given by eq. (35) when $\mu \neq \nu$.

Let us first calculate the effect due to the stochastic forces of the background. This term could cause an energy flux responsible for the satellite deacceleration. Eq. (35) with $\mu = S, \nu = B$ and neglecting the heating term, gives:

$$\left\langle \left(\frac{dE^{B}_{S,tot}}{dt} \right)_{\delta t} \right\rangle =$$
$$= (\beta_S - \beta_B) \sum_{i_S} \sum_{j_S} \int_{-\infty}^0 ds \boldsymbol{v}_{i_S,0} C^{BB}_{i_S j_S}(s) \boldsymbol{v}_{j_S,0}, \quad (37)$$

where the correlation matrix is given by eq. (A10) with $\alpha = B$ and $\gamma = \delta = S$. In the case of a point-like satellite with $m_S \gg m_B$, $\beta_S \ll \beta_B$ and then the satellite loses energy to the background. In our case, however, it is not excluded in principle that $\beta_S > \beta_B$ and then the energy would flow from the background to the satellite.

The integral over ds in eq. (37) can be calculated taking into account eq. (A10) and the equality:

$$\int_{-\infty}^0 ds \frac{\boldsymbol{A} + \boldsymbol{V}s}{\mid \boldsymbol{A} + \boldsymbol{V}s \mid^3} = \frac{\boldsymbol{A} \cdot \boldsymbol{V} + AV}{V^2 A_\perp^2} \left[\frac{\boldsymbol{A}}{A} - \frac{\boldsymbol{V}}{V} \right] \quad (38)$$

where $\boldsymbol{A}_\perp$ is the projection of vector $\boldsymbol{A}$ on the plane normal to vector $\boldsymbol{V}$. In our case either $\boldsymbol{V} \equiv \boldsymbol{v}_{j_S} - \boldsymbol{v}_{h_B}$ or $\boldsymbol{V} \equiv \boldsymbol{v}_{j_S}$, (see eq. (A10)), and $\boldsymbol{A} \equiv \boldsymbol{r}_{j_S} - \boldsymbol{r}_{h_B}$ or $\boldsymbol{A} \equiv \boldsymbol{r}_{j_S} - \boldsymbol{r}'$. We obtain:

$$\left\langle \left(\frac{dE^{B}_{S,tot}}{dt} \right)_{\delta t} \right\rangle = G^2 m_B^2 m_S^2 (\beta_S - \beta_B) \frac{N_B'}{N_B}$$
$$\times \sum_{i_S} \sum_{j_S} \Bigg[\sum_{h_B} \left\{ \frac{(\boldsymbol{r}_{i_S} - \boldsymbol{r}_{h_B}) \cdot \boldsymbol{v}_{i_S}}{\mid \boldsymbol{r}_{i_S} - \boldsymbol{r}_{h_B} \mid^3} \right\}$$
$$\times \Bigg\{ \frac{[\boldsymbol{v}_{j_S} \cdot (\boldsymbol{r}_{j_S} - \boldsymbol{r}_{h_B})][(\boldsymbol{r}_{j_S} - \boldsymbol{r}_{h_B}) \cdot (\boldsymbol{v}_{j_S} - \boldsymbol{v}_{h_B})]}{\mid \boldsymbol{r}_{j_S} - \boldsymbol{r}_{h_B} \mid\mid \boldsymbol{v}_{j_S} - \boldsymbol{v}_{h_B} \mid^2 \mid (\boldsymbol{r}_{j_S} - \boldsymbol{r}_{h_B})_\perp \mid^2}$$
$$- \frac{\mid \boldsymbol{r}_{j_S} - \boldsymbol{r}_{h_B} \mid^2 (\boldsymbol{v}_{j_S} - \boldsymbol{v}_{h_B}) \cdot \boldsymbol{v}_{j_S}}{\mid \boldsymbol{r}_{j_S} - \boldsymbol{r}_{h_B} \mid\mid \boldsymbol{v}_{j_S} - \boldsymbol{v}_{h_B} \mid^2 \mid (\boldsymbol{r}_{j_S} - \boldsymbol{r}_{h_B})_\perp \mid^2}$$
$$+ \frac{\boldsymbol{v}_{j_S} \cdot (\boldsymbol{r}_{j_S} - \boldsymbol{r}_{h_B})_\perp}{\mid \boldsymbol{v}_{j_S} - \boldsymbol{v}_{h_B} \mid\mid (\boldsymbol{r}_{j_S} - \boldsymbol{r}_{h_B})_\perp \mid^2} \Bigg\}$$
$$+ \frac{1}{N_B} \int d\boldsymbol{r} d\boldsymbol{r}' n_B(\boldsymbol{r}) n_B(\boldsymbol{r}') \frac{[(\boldsymbol{r}_{i_S} - \boldsymbol{r}) \cdot \boldsymbol{v}_{i_S}]}{\mid \boldsymbol{r}_{i_S} - \boldsymbol{r} \mid^3 \mid \boldsymbol{r}_{j_S} - \boldsymbol{r}' \mid} \Bigg] \quad (39)$$

where the sum on a generic subindex, i_α, means N_α times the average on velocity and positions of class α particles, and is carried out by means of the distribution function given in eq. (28).

To proceed further, we recall that the velocity distribution of background particles is assumed to be Maxwellian with zero mean and dispersion σ_B, and the velocity distribution of the satellite particles has mean equal to the satellite center of mass velocity, $\boldsymbol{v}_{CMS}$, and dispersion σ_S. The integration over $d\boldsymbol{v}_{h_B}$ gives:

$$\left\langle \left(\frac{dE^{B}_{S,tot}}{dt} \right)_{\delta t} \right\rangle = G^2 m_B^2 m_S^2 (\beta_S - \beta_B)$$

$$\times \sum_{i_S} \sum_{j_S} \left[\int d\boldsymbol{r}_{h_B} n_B(\boldsymbol{r}_{h_B}) \left\{ \frac{(\boldsymbol{r}_{i_S} - \boldsymbol{r}_{h_B}) \cdot \boldsymbol{v}_{i_S}}{\mid \boldsymbol{r}_{i_S} - \boldsymbol{r}_{h_B} \mid^3} \right\} \right.$$
$$\times \left\{ \frac{\exp(\gamma_{j_S}^2 - y_{j_S}^2)\mathrm{erf}(\gamma_{j_S}) - 1}{\mid \boldsymbol{r}_{j_S} - \boldsymbol{r}_{h_B} \mid} \right\}$$
$$\left. + \frac{1}{N_B} \int d\boldsymbol{r} d\boldsymbol{r}' n_B(\boldsymbol{r}) n_B(\boldsymbol{r}') \frac{[(\boldsymbol{r}_{i_S} - \boldsymbol{r}) \cdot \boldsymbol{v}_{i_S}]}{\mid \boldsymbol{r}_{i_S} - \boldsymbol{r} \mid^3 \mid \boldsymbol{r}_{j_S} - \boldsymbol{r}' \mid} \right] \quad (40)$$

where erf stands for the error function, $\boldsymbol{y}_{j_S} \equiv \frac{\boldsymbol{v}_{j_S}}{\sqrt{2}\sigma_B}$, $\gamma_{j_S} \equiv \boldsymbol{y}_{j_S} \cdot \boldsymbol{\zeta}$ with $\boldsymbol{\zeta} \equiv \frac{(\boldsymbol{r}_{j_S} - \boldsymbol{r}_{h_B})}{|\boldsymbol{r}_{j_S} - \boldsymbol{r}_{h_B}|}$ and taking $\frac{N_B^{'}}{N_B} = 1$.

The integration over $d\boldsymbol{v}_{j_S}$ involves the integral:

$$I = \int d\boldsymbol{v}_{j_S} f_M(\boldsymbol{v}_{j_S}) \exp(\gamma_{j_S}^2 - y_{j_S}^2)\mathrm{erf}(\gamma_{j_S}). \quad (41)$$

In Appendix B we show that

$$I = \frac{\sigma_B^2}{\sigma_T^2} \exp(\alpha_{CMS}^2 - x_{CMS}^2)\mathrm{erf}(\alpha_{CMS}), \quad (42)$$

where $\sigma_T^2 = \sigma_B^2 + \sigma_S^2$, $\boldsymbol{x}_{CMS} \equiv \frac{\boldsymbol{v}_{CMS}}{\sqrt{2}\sigma_T}$ and $\alpha_{CMS} \equiv \boldsymbol{x}_{CMS} \cdot \boldsymbol{\zeta}$. The integration over $d\boldsymbol{v}_{i_S}$ is trivial recalling that $\boldsymbol{v}_{i_S} = \boldsymbol{v}_{CMS} + \boldsymbol{u}_{i_S}$ and that for a Maxwellian distribution the contribution of $\boldsymbol{u}_{i_S}$ to the first moments vanishes. Taking $m_\alpha n_\alpha = \rho_\alpha$, we finally get:

$$\left\langle \left(\frac{dE_{S,tot}^B}{dt} \right)_{\delta t} \right\rangle = \left(\frac{m_B \sigma_B^2}{m_S \sigma_S^2} - 1 \right) \frac{G^2}{\sigma_B^2} \int d\boldsymbol{r}_{i_S} d\boldsymbol{r}_{j_S} \rho_S(\boldsymbol{r}_{i_S}) \rho_S(\boldsymbol{r}_{j_S})$$
$$\times \left[\int d\boldsymbol{r}_{h_B} \rho_B(\boldsymbol{r}_{h_B}) \left\{ \frac{(\boldsymbol{r}_{i_S} - \boldsymbol{r}_{h_B}) \cdot \boldsymbol{v}_{CMS}}{\mid \boldsymbol{r}_{i_S} - \boldsymbol{r}_{h_B} \mid^4} \right\} \right.$$
$$\times \left\{ \frac{\sigma_B^2}{\sigma_T^2} exp(\alpha_{CMS}^2 - x_{CMS}^2)\mathrm{erf}(\alpha_{CMS}) - 1 \right\}$$
$$\left. + \frac{1}{N_B} \int d\boldsymbol{r} d\boldsymbol{r}' \rho_B(\boldsymbol{r}) n_B(\boldsymbol{r}') \frac{[(\boldsymbol{r}_{i_S} - \boldsymbol{r}) \cdot \boldsymbol{v}_{CMS}]}{\mid \boldsymbol{r}_{i_S} - \boldsymbol{r} \mid^3 \mid \boldsymbol{r}_{j_S} - \boldsymbol{r}' \mid} \right] (43)$$

A specification of the density distribution of both the satellite and the background is needed in order to carry out the integrals over the space variables.

Next we calculate the variation of the background energy caused by the stochastic forces of the satellite. It can be obtained from eq.(35) with $\mu = B$ and $\nu = S$, neglecting the heating term:

$$\left\langle \left(\frac{dE_{B,tot}^S}{dt} \right)_{\delta t} \right\rangle =$$
$$= (\beta_B - \beta_S) \sum_{i_B} \sum_{j_B} \int_{-\infty}^{0} ds \boldsymbol{v}_{i_B,0} C_{i_B j_B}^{SS}(s) \boldsymbol{v}_{j_B,0}. \quad (44)$$

For a background at rest with a Maxwellian velocity distribution function, this energy rate vanishes when one performs the integration over $d\boldsymbol{v}_{i_B}$. We conclude that the effect of fluctuating forces of the satellite acting on the background only heat it, and have no effect on a variation of the orbital energy of either the satellite or the background.

The total instantaneous energy flow between the satellite and the background is given by the difference between the l.h.s. of the eqs. (43) and (44) (see eq. (32)). It can be written in terms of the rate of change of the satellite orbital and internal energies as:

$$\left\langle \left(\frac{dE_{tot}^S}{dt} \right)_{\delta t} \right\rangle = \left\langle \left(\frac{dE_{orb}^S}{dt} \right)_{\delta t} \right\rangle + \left\langle \left(\frac{dE_{in}^S}{dt} \right)_{\delta t} \right\rangle$$
$$= \left\langle \left(\frac{dE_{S,tot}^B}{dt} \right)_{\delta t} \right\rangle \quad (45)$$

where E_{orb}^S and E_{in}^S stand for the orbital and internal energy of the satellite, respectively, and the second equality results from the zero value of the flow given by eq. (44). The rate of change of the satellite internal energy can be calculated with the help of eq. (24) with $Q = \frac{dE_{in}^S}{dt}$ or:

$$Q = m_S \sum_{i_S} \boldsymbol{u}_{i_S} \frac{d}{dt} \boldsymbol{u}_{i_S}, \quad (46)$$

where $\boldsymbol{u}_{i_S} = \boldsymbol{v}_{i_S} - \boldsymbol{v}_{CMS}$ is the velocity of i_S particles with respect to its center of mass. This gives an expression similar to eq. (43), except that now $\boldsymbol{v}_{CMS}$ takes a zero value, so that

$$\left\langle \left(\frac{dE_{in}^S}{dt} \right)_{\delta t} \right\rangle = 0, \quad (47)$$

and then eq. (43) gives, at second order in the fluctuations, the rate of change of the satellite *orbital* energy.

3.2 The self-interaction energies of the background and the satellite

When $\mu = \nu = B$ or S, eq. (35) describes the instantaneous energy variation rate of class μ particles due to the stochastic forces caused by μ particles themselves.

According to the eq. (32) these energies do not play any role in the energy change of the subsystems S or B. They only represent the energy change of an individual particle (see eq. (31)).

The self-interaction energy of the background is easily obtained from eq. (35) with $\mu = \nu = B$:

$$\left\langle \left(\frac{dE_{B,tot}^B}{dt} \right)_{\delta t} \right\rangle =$$
$$= (\beta_S - \beta_B) \sum_{i_B} \sum_{j_S} \int_{-\infty}^{0} ds \boldsymbol{v}_{i_B,0} C_{i_B j_S}^{BB}(s) \boldsymbol{v}_{j_S,0}. \quad (48)$$

Again, the integration over $d\boldsymbol{v}_{i_B}$ makes it vanish for a background at rest with an isotropic velocity distribution function: because no translation energy is available, the autointeraction results only in a slow heating of the background.

Regarding the autointeraction energy of the satellite, eq. (35) gives:

$$\left\langle \left(\frac{dE_{S,tot}^S}{dt} \right)_{\delta t} \right\rangle =$$
$$= (\beta_B - \beta_S) \sum_{i_S} \sum_{j_B} \int_{-\infty}^{0} ds \boldsymbol{v}_{i_S,0} C_{i_S j_B}^{SS}(s) \boldsymbol{v}_{j_B,0}. \quad (49)$$

The correlation matrix is given by eq. (A10) with $\alpha = S, \gamma = S$ and $\delta = B$. Substituting the expression for $C_{i_S j_B}^{SS}(s)$ in eq. (49), and performing the integral over ds with the help of eq. (38), we obtain:

$$\left\langle\left(\frac{dE^S_{S,tot}}{dt}\right)_{\delta t}\right\rangle = G^2 m_B^2 m_S^2(\beta_B-\beta_S)\frac{N_S^{'}}{N_S}$$
$$\times\sum_{i_S}\sum_{j_B}\left[\sum_{h_S}\left\{\frac{(\boldsymbol{r}_{i_S}-\boldsymbol{r}_{h_S})\cdot\boldsymbol{v}_{i_S}}{\mid\boldsymbol{r}_{i_S}-\boldsymbol{r}_{h_S}\mid^3}\right\}\right.$$
$$\times\left\{\frac{[\boldsymbol{v}_{j_B}\cdot(\boldsymbol{r}_{j_B}-\boldsymbol{r}_{h_S})][(\boldsymbol{r}_{j_B}-\boldsymbol{r}_{h_S})\cdot(\boldsymbol{v}_{j_B}-\boldsymbol{v}_{h_S})]}{\mid\boldsymbol{r}_{j_B}-\boldsymbol{r}_{h_S}\mid\mid\boldsymbol{v}_{j_B}-\boldsymbol{v}_{h_S}\mid^2\mid(\boldsymbol{r}_{j_B}-\boldsymbol{r}_{h_S})_\perp\mid^2}\right.$$
$$-\frac{\mid\boldsymbol{r}_{j_B}-\boldsymbol{r}_{h_S}\mid^2(\boldsymbol{v}_{j_B}-\boldsymbol{v}_{h_S})\cdot\boldsymbol{v}_{j_B}}{\mid\boldsymbol{r}_{j_B}-\boldsymbol{r}_{h_S}\mid\mid\boldsymbol{v}_{j_B}-\boldsymbol{v}_{h_S}\mid^2\mid(\boldsymbol{r}_{j_B}-\boldsymbol{r}_{h_S})_\perp\mid^2}$$
$$\left.+\frac{\boldsymbol{v}_{j_B}\cdot(\boldsymbol{r}_{j_B}-\boldsymbol{r}_{h_S})_\perp}{\mid\boldsymbol{v}_{j_B}-\boldsymbol{v}_{h_S}\mid\mid(\boldsymbol{r}_{j_B}-\boldsymbol{r}_{h_S})_\perp\mid^2}\right\}$$
$$\left.+\frac{1}{N_S}\int d\boldsymbol{r}d\boldsymbol{r}' n_S(\boldsymbol{r})n_S(\boldsymbol{r}')\frac{[(\boldsymbol{r}_{i_S}-\boldsymbol{r})\cdot\boldsymbol{v}_{i_S}]}{\mid\boldsymbol{r}_{i_S}-\boldsymbol{r}\mid^3\mid\boldsymbol{r}_{j_B}-\boldsymbol{r}'\mid}\right].\quad(50)$$

The integrations over velocities can be carried out following the same steps as in the previous section and we finally have:

$$\left\langle\left(\frac{dE^S_{S,tot}}{dt}\right)_{\delta t}\right\rangle = \left(\frac{m_S\sigma_S^2}{m_B\sigma_B^2}-1\right)\frac{G^2}{\sigma_S^2}\int d\boldsymbol{r}_{i_S}d\boldsymbol{r}_{j_B}\rho_S(\boldsymbol{r}_{i_S})\rho_B(\boldsymbol{r}_{j_B})$$
$$\times\left[-\frac{\sigma_B^2}{\sigma_T^2}\int d\boldsymbol{r}_{h_S}\rho_S(\boldsymbol{r}_{h_S})\left\{\frac{(\boldsymbol{r}_{i_S}-\boldsymbol{r}_{h_S})\cdot\boldsymbol{v}_{CMS}}{\mid\boldsymbol{r}_{i_S}-\boldsymbol{r}_{h_S}\mid^3}\right\}\right.$$
$$\times\left\{\frac{exp(\alpha^2_{CMS}-x^2_{CMS})\mathrm{erf}(\alpha_{CMS})}{\mid\boldsymbol{r}_{j_B}-\boldsymbol{r}_{h_S}\mid}\right\}$$
$$\left.+\frac{1}{N_S}\int d\boldsymbol{r}d\boldsymbol{r}'\rho_S(\boldsymbol{r})n_S(\boldsymbol{r}')\frac{[(\boldsymbol{r}_{i_S}-\boldsymbol{r})\cdot\boldsymbol{v}_{CMS}]}{\mid\boldsymbol{r}_{i_S}-\boldsymbol{r}\mid^3\mid\boldsymbol{r}_{j_B}-\boldsymbol{r}'\mid}\right]\quad(51)$$

where $\alpha_{CMS}\equiv\boldsymbol{x}_{CMS}\cdot\frac{(\boldsymbol{r}_{h_S}-\boldsymbol{r}_{j_B})}{|\boldsymbol{r}_{h_S}-\boldsymbol{r}_{j_B}|}$.

4 PARTICULAR LIMITS

4.1 Point-mass satellites

In eq. (43) it is implicitly assumed that the distance between a generic satellite particle and a generic background particle cannot be smaller than a scale, d_{min}, that is, that there exists a minimum effective impact parameter. As the main contribution to the integrals appearing in this expression comes from small $\mid\boldsymbol{r}_S-\boldsymbol{r}_B\mid$ values ($\boldsymbol{r}_S$ and $\boldsymbol{r}_B$ are generic satellite and background particle positions), an accurate determination of the d_{min} scale is a crucial point when studying dynamical friction. This scale, however, cannot be determined by the present approach to this problem. There exist in literature several estimates of d_{min} for specific situations (White 1976; Bontekoe & van Albada 1987). White (1976) in fact shows that for spherical symmetric satellites, it is a good approximation to consider them as point-like systems with a cut-off in their distances to background particles, and that for the case of satellites with a King mass distribution (King 1966), this cut-off is given by $d_{min}\simeq\frac{R_t}{5}$, where R_t is the satellite tidal radius. A point mass satellite has a density profile given by $\rho_S(\boldsymbol{r}_S)=M_S\delta(\boldsymbol{r}_S-\boldsymbol{r}_{CMS})$, where $\delta(\boldsymbol{r}_S-\boldsymbol{r}_{CMS})$ is the delta function in three dimensions and $\boldsymbol{r}_{CMS}$ is the position of the satellite center-of-mass. With this value of $\rho_S(\boldsymbol{r}_S)$, eq. (43) gives in the point mass limit:

$$\left\langle\left(\frac{dE^S_{orb}}{dt}\right)_{\delta t}\right\rangle = \left(\frac{m_B\sigma_B^2}{m_S\sigma_S^2}-1\right)\frac{G^2M_S^2}{\sigma_B^2}$$
$$\times\left[\int d\boldsymbol{r}_{h_B}\rho_B(\boldsymbol{r}_{h_B})\Theta(\boldsymbol{r}_{h_B})\frac{(\boldsymbol{r}_{CMS}-\boldsymbol{r}_{h_B})\cdot\boldsymbol{v}_{CMS}}{\mid\boldsymbol{r}_{CMS}-\boldsymbol{r}_{h_B}\mid^4}\right.$$
$$\times\left\{\frac{\sigma_B^2}{\sigma_T^2}exp(\alpha^2_{CMS}-x^2_{CMS})\mathrm{erf}(\alpha_{CMS})-1\right\}$$
$$+\frac{1}{N_B}\int d\boldsymbol{r}d\boldsymbol{r}'\rho_B(\boldsymbol{r})n_B(\boldsymbol{r}')\Theta(\boldsymbol{r})\Theta(\boldsymbol{r}')$$
$$\left.\times\frac{[(\boldsymbol{r}_{CMS}-\boldsymbol{r})\cdot\boldsymbol{v}_{CMS}]}{\mid\boldsymbol{r}_{CMS}-\boldsymbol{r}\mid^3\mid\boldsymbol{r}_{CMS}-\boldsymbol{r}'\mid}\right]\quad(52)$$

where now $\alpha_{CMS}\equiv\boldsymbol{x}_{CMS}\cdot\frac{(\boldsymbol{r}_{CMS}-\boldsymbol{r}_{h_B})}{|\boldsymbol{r}_{CMS}-\boldsymbol{r}_{h_B}|}$ and $\Theta(\boldsymbol{r})\equiv\Theta(\mid\boldsymbol{r}_{CMS}-\boldsymbol{r}\mid-d_{min})$ is the step function.

4.2 Homogeneous backgrounds. The Chandrasekhar limit

When the background density is uniform, we can set $\rho_B(\boldsymbol{r}_B)=\rho_{B,0}$ and $\boldsymbol{r}_{CMS}=0$ in eq. (52) and then it becomes:

$$\left\langle\left(\frac{dE^S_{orb}}{dt}\right)_{\delta t}\right\rangle = \left(\frac{m_B\sigma_B^2}{m_S\sigma_S^2}-1\right)\frac{4\pi G^2M_S^2\rho_{B,0}\ln\Lambda}{v_{CMS}}$$
$$\times[\mathrm{erf}(x_{CMS})-\frac{2}{\sqrt{\pi}}x_{CMS}\exp(-x^2_{CMS})].\quad(53)$$

This recovers the Chandrasekhar formula for the motion of a massive test particle in an homogeneous background, except for the factor containing the temperature ratio (which vanishes in the Chandrasekhar formula because $m_S\gg m_B$ in this limit), and, now, $\sigma_T=(\sigma_S^2+\sigma_B^2)^{1/2}$ at the place of the background velocity dispersion.

In the Figure 1 we plot the ratio, R, of satellite energy loss in the Chandrasekhar limit with $\sigma_S=0$ and with $\sigma_S\neq 0$, assuming that $T_B/T_S=0$. As can be seen in this Figure, the effect of a non zero σ_S increases with increasing σ_S and is more important at low v_{CMS}. The Chandrasekhar formula always overestimates the dynamical friction force, and in some situations the effect of neglecting the satellite velocity dispersion could cause an error as high as an order of magnitude.

5 SUMMARY AND DISCUSSION

We have derived an expression giving the orbital energy exchange due to dynamical friction experienced by an extended body, composed of N_S bound particles endowed with a velocity spectrum, as it moves interacting with a non homogeneous discrete background. It has been assumed that both, the satellite and the background, have Maxwellian velocity distributions and that the background is static.

Self-interactions of both satellite and background particles have been taken into account. This results in no effect on their energy exchange.

Heating terms appear in quite a natural way in our approach both due to interactions among particles of the

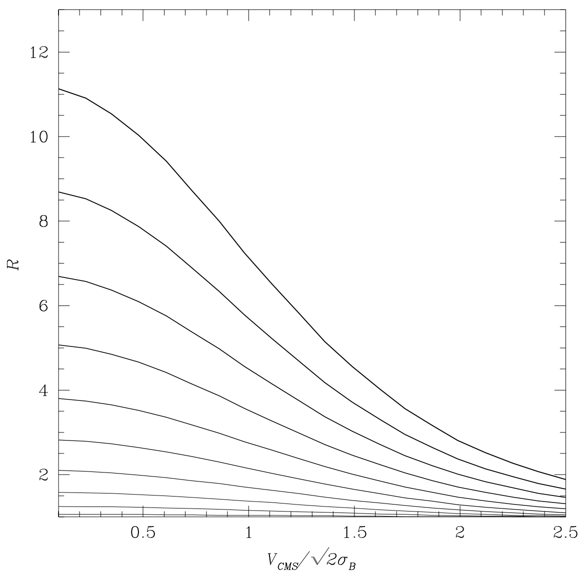

Figure 1. Ratio, R, of satellite energy loss in the Chandrasekhar limit with $\sigma_S = 0$ and with $\sigma_S \neq 0$, assuming that $T_B/T_S = 0$. Lines correspond to different σ_S/σ_B ratios from 0.2 to 2.0 (ratio increases as the thickness of the line increses)

same kind or of different kind. They are a factor of $1/N$ smaller than the orbital effects.

In the point-like satellite limit (or small as compared with the background size) and constant density background, we obtain an expression that recovers, in the limit $\sigma_S/\sigma_B \to 0$ and $T_S/T_B \to \infty$, Chandrasekhar's dynamical friction formula (eq. (1)). Its comparison with the energy loss given by eq. (1) allows for a quantification of the effects of having a non zero σ_S. It has been found out that the energy loss is always smaller in this case, and that the difference can be up to about an order of magnitude for slow satellites as compared with σ_B.

In deriving eq. (35) we have considered time intervals, δt, that are short as compared with the time scale for the variation of the particle velocities and positions. This allows us to neglect the effects of the smooth gravitational potential gradients. However, in order to carry out a precise calculation of dynamical friction, the whole history of the system from an early enough time and the interactions along the entire satellite trajectory should have been taken into account. This would have made the problem extremely difficult to solve. Instead, taking only a finite δt, means that interactions with distant particles have not been accurately considered. Nevertheless, we recall that the contribution of particles to dynamical friction quickly decreases with distance, so that this neglect should not result in major consequences.

The bound of δt has also another consequence: this approach is unable to describe slowly accumulating effects on dynamical friction (Kalnajs 1972) or the effect of reversible dynamical feedback (e.g. Tremaine & Weinberg 1984), because they arise as a consequence of the periodic motion of the satellite after many revolutions.

Despite these shortcomings, the extension of the fluctuation-dissipation approach to dynamical friction presented in this paper, has resulted in the derivation of a formula that takes into account the space and velocity structure of the satellite. This represents a common situation in many astrophysical processes and, as we have shown, might have important quantitative consequences in the setting-up of timescales for these processes.

ACKNOWLEDGMENTS

We thank Drs. G. González-Casado and E. Salvador-Solé for interesting informations. M.A. Gómez-Flechoso was supported by the Dirección General de Investigación Científica y Técnica (DGICYT, Spain) through a fellowship.

The DGICYT also supported in part this work , grants AEN93-0673 and PB93-0252.

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APPENDIX A: THE CORRELATION MATRIX

The correlation matrix is defined in eq. (36) with the fluctuating forces given by eq. (10). Each matrix is an object with four index, and each index takes on two different values (B and S). This makes sixteen different possibilities, corresponding to the tensorial product of four different possibilities for the $\boldsymbol{F}^{\alpha}_{i_\gamma}(t)$ forces. Writing the Fourier transform of $\mid \boldsymbol{r} - \boldsymbol{r}' \mid^{-1}$ and $n_\alpha(\boldsymbol{r})$ with respect to $\boldsymbol{r}$, the convolution theorem and eq. (10) imply that:

$$\boldsymbol{F}^{\alpha}_{i_\gamma}(\boldsymbol{r}_{i_\gamma},t) = -i\frac{Gm_\alpha m_\gamma}{2\pi^2}\int\frac{d\boldsymbol{k}}{k^2}\boldsymbol{k}\exp[i\boldsymbol{k}\cdot\boldsymbol{r}_{i_\gamma}(t)] \times\left[\sum_{g_\alpha\neq i_\gamma}\exp[-i\boldsymbol{k}\cdot\boldsymbol{r}_{g_\alpha}(t)] - \frac{N'_\alpha}{N_\alpha}n^\alpha_k\right], \tag{A1}$$

where

$$n^\alpha_k \equiv \int d\boldsymbol{r} n_\alpha(\boldsymbol{r}) exp[-i\boldsymbol{k}\cdot\boldsymbol{r}]. \tag{A2}$$

With this definition

$$\langle\exp[-i\boldsymbol{k}\cdot\boldsymbol{r}_{i_\alpha}]\rangle_0 = \frac{n^\alpha_k}{N_\alpha} \tag{A3}$$

and then the average of the stochastic forces, eq. (A1), in the unperturbed state vanishes, as required.

Inserting eq. (A1) in eq. (36), we get:

$$C^{\alpha\beta}_{i_\gamma j_\delta}(s) = -\frac{G^2 m_\alpha m_\beta m_\gamma m_\delta}{4\pi^4} \times\int\frac{d\boldsymbol{k}}{k^2}\frac{d\boldsymbol{l}}{l^2}\boldsymbol{k}\otimes\boldsymbol{l}\exp[i(\boldsymbol{k}\cdot\boldsymbol{r}_{i_\gamma}(0)+\boldsymbol{l}\cdot\boldsymbol{r}_{j_\delta}(s))] \times\left\langle\left[\sum_{g_\alpha\neq i_\gamma}\exp[-i\boldsymbol{k}\cdot\boldsymbol{r}_{g_\alpha}(0)] - \frac{N'_\alpha}{N_\alpha}n^\alpha_k\right] \times\left[\sum_{h_\beta\neq j_\delta}\exp[-i\boldsymbol{l}\cdot\boldsymbol{r}_{h_\beta}(s)] - \frac{N'_\beta}{N_\beta}n^\beta_l\right]\right\rangle_0 \tag{A4}$$

and eq. (A3) now gives:

$$C^{\alpha\beta}_{i_\gamma j_\delta}(s) = -\frac{G^2 m_\alpha m_\beta m_\gamma m_\delta}{4\pi^4} \times\int\frac{d\boldsymbol{k}}{k^2}\frac{d\boldsymbol{l}}{l^2}\boldsymbol{k}\otimes\boldsymbol{l}\exp[i(\boldsymbol{k}\cdot\boldsymbol{r}_{i_\gamma}(0)+\boldsymbol{l}\cdot\boldsymbol{r}_{j_\delta}(s))] \times\left[\sum_{g_\alpha\neq i_\gamma}\sum_{h_\beta\neq j_\delta}E_{kl}(g_\alpha,h_\beta,s) - \frac{N'_\alpha}{N_\alpha}\frac{N'_\beta}{N_\beta}n^\alpha_k n^\beta_l\right] \tag{A5}$$

where

$$E_{kl}(g_\alpha,h_\beta,s) \equiv \langle\exp\{-i[\boldsymbol{k}\cdot\boldsymbol{r}_{g_\alpha}(0)+\boldsymbol{l}\cdot\boldsymbol{r}_{h_\beta}(s)]\}\rangle_0. \tag{A6}$$

It is assumed that particles are uncorrelated in the unperturbed state. Then, if $\alpha\neq\beta$, necessarily $g_\alpha\neq h_\beta$ and the average in eq. (A6) factorizes, as corresponding to uncorrelated variables, giving $C^{\alpha\beta}_{i_\gamma j_\delta}(s)=0$. The average also factorizes for different particles belonging to the same particle class, i.e., when $\alpha=\beta$ but $g_\alpha\neq h_\alpha$. There are $N'_\alpha(N'_\alpha-1)$ such terms, and each of them has the value $\frac{n^\alpha_k n^\alpha_l}{N^2_\alpha}$. When $g_\alpha = h_\alpha$, that is, when we consider the same particle, the average does not factorize anymore. Taking this considerations into account, the correlation matrix becomes:

$$C^{\alpha\alpha}_{i_\gamma j_\delta}(s) = -\frac{G^2 m^2_\alpha m_\gamma m_\delta}{4\pi^4} \times\int\frac{d\boldsymbol{k}}{k^2}\frac{d\boldsymbol{l}}{l^2}\boldsymbol{k}\otimes\boldsymbol{l}\exp[i(\boldsymbol{k}\cdot\boldsymbol{r}_{i_\gamma}(0)+\boldsymbol{l}\cdot\boldsymbol{r}_{j_\delta}(s))] \times\left[N'_\alpha E_{kl}(g_\alpha=h_\alpha,s) - N'_\alpha\frac{n^\alpha_k n^\alpha_l}{N^2_\alpha}\right]. \tag{A7}$$

The position of a generic particle at time s satisfying (19) can be written as:

$$\boldsymbol{r}_{j_\alpha}(s) = \boldsymbol{r}_{j_\alpha,0} + \boldsymbol{v}_{j_\alpha,0}s \tag{A8}$$

An integration of eq. (11) would give four more terms corresponding to the gravitational acceleration and the fluctuating forces. The acceleration terms are negligible when compared with $\boldsymbol{v}_{j_\alpha,0}s$ (see eq. (19)) and the fluctuating forces would give rise to third order terms.

Once this expression for the particle positions at time s is substituted in eq. (A7), the integrals over $d\boldsymbol{k}$ and $d\boldsymbol{l}$ can be easily calculated taking the gradient of the Fourier representation for the Green's function of the Laplace equation in three dimensions:

$$\int\frac{d\boldsymbol{k}}{k^2}\boldsymbol{k}\exp[i\boldsymbol{k}\cdot\boldsymbol{A}] = i\frac{2\pi\boldsymbol{A}}{A^3}. \tag{A9}$$

Finally, combining eqs. (A2), (A6), (A7), (A8) and (A9) the correlation matrix reads:

$$C^{\alpha\alpha}_{i_\gamma j_\delta}(s) = G^2 m^2_\alpha m_\gamma m_\delta\frac{N'_\alpha}{N_\alpha} \times\left[\sum_{h_\alpha}\frac{(\boldsymbol{r}_{i_\gamma}-\boldsymbol{r}_{h_\alpha})}{|\boldsymbol{r}_{i_\gamma}-\boldsymbol{r}_{h_\alpha}|^3}\otimes\frac{[\boldsymbol{r}_{j_\delta}-\boldsymbol{r}_{h_\alpha}+s(\boldsymbol{v}_{j_\delta}-\boldsymbol{v}_{h_\alpha})]}{|\boldsymbol{r}_{j_\delta}-\boldsymbol{r}_{h_\alpha}+s(\boldsymbol{v}_{j_\delta}-\boldsymbol{v}_{h_\alpha})|^3} - \frac{1}{N_\alpha}\int d\boldsymbol{r}d\boldsymbol{r}' n_\alpha(\boldsymbol{r})n_\alpha(\boldsymbol{r}') \times\frac{(\boldsymbol{r}_{i_\gamma}-\boldsymbol{r})}{|\,\boldsymbol{r}_{i_\gamma}-\boldsymbol{r}\,|^3}\otimes\frac{(\boldsymbol{r}_{j_\delta}-\boldsymbol{r}'+s\boldsymbol{v}_{j_\delta})}{|\,\boldsymbol{r}_{j_\delta}-\boldsymbol{r}'+s\boldsymbol{v}_{j_\delta}\,|^3}\right]. \tag{A10}$$

The 0 subindex have been dropped from the $\boldsymbol{r}$ and $\boldsymbol{v}$ vectors.

APPENDIX B:

In this appendix we calculate the integral

$$I = \int d\boldsymbol{v} f^S_M(\boldsymbol{v})\exp\left[\gamma^2-y^2\right]\mathrm{erf}(\gamma), \tag{B1}$$

where $\boldsymbol{v}$ is the velocity of class S particles, that is

$$\boldsymbol{v} = \boldsymbol{v}_{CMS} + \boldsymbol{u}, \tag{B2}$$

whose distribution function, f^S_M, is a Maxwellian isotropic in $\boldsymbol{u}$

$$f^S_M(\boldsymbol{v}) = \frac{1}{(2\pi\sigma^2_S)^{3/2}}\exp\left[-\frac{(\boldsymbol{v}-\boldsymbol{v}_{CMS})^2}{2\sigma^2_S}\right]. \tag{B3}$$

In the last equation $\boldsymbol{y}\equiv\boldsymbol{v}/\sqrt{2}\sigma_B$, $\gamma\equiv\boldsymbol{y}\cdot\boldsymbol{\zeta}$, and $\boldsymbol{\zeta}$ is a unit constant vector. Expression (B2) implies:

$$\gamma \equiv \gamma_{CMS} + \gamma_u \tag{B4}$$

where

$$\gamma_{CMS} = \frac{\boldsymbol{v}_{CMS}}{\sqrt{2}\sigma_B}\boldsymbol{\zeta}$$

and

$$\gamma_u = \frac{\boldsymbol{u}_{CMS}}{\sqrt{2}\sigma_B}\boldsymbol{\zeta}$$

The definition of the error function and equation (B4) imply that:

$$\mathrm{erf}(\gamma) = \mathrm{erf}(\gamma_{CMS}) + \frac{2}{\sqrt{\pi}}\int_0^{\gamma_u} dt \exp\left[-(t+\gamma_{CMS})^2\right]. \quad \text{(B5)}$$

We now write I (eq. (B1)) as an integral with respect to the relative velocities $\boldsymbol{u}$. Taking $\boldsymbol{\zeta} = (0,0,1)$ we obtain:

$$I = \frac{\exp\left(\alpha_{CMS}^2 - x_{CMS}^2\right)}{(2\pi\sigma_S^2)^{3/2}} \int du_x \exp\left[-(Au_x + B_x)^2\right]$$
$$\times \int du_y \exp\left[-(Au_y + B_y)^2\right] \int du_z \exp\left[-\left(\frac{u_z}{\sqrt{2}\sigma_S}\right)^2\right]$$
$$\times \left[\mathrm{erf}(\gamma_{CMS}) + \frac{2}{\sqrt{\pi}}\int_0^{\gamma_u} dt \exp\left[-(t+\gamma_{CMS})^2\right]\right] \quad \text{(B6)}$$

where $\boldsymbol{x}_{CMS} \equiv \boldsymbol{v}_{CMS}/\sqrt{2}\sigma_T$, $\alpha_{CMS} \equiv \boldsymbol{x}_{CMS} \cdot \boldsymbol{\zeta}$, $\sigma_T^2 \equiv \sigma_B^2 + \sigma_S^2$, $A \equiv \sigma_T/\sqrt{2}\sigma_S\sigma_B$, $B_i \equiv \frac{\sigma_S}{\sigma_B} x_{CMS,i}$, with $i = x, y$, and $\gamma_u = u_z/\sqrt{2}\pi\sigma_B$. The integrals over du_x and du_y can be easily performed and they give:

$$\left[\frac{\sqrt{\pi}}{A}\mathrm{erf}(\infty)\right]^2 = \frac{\pi}{A^2}$$

The integral can now be written as:

$$I = \frac{\sigma_H^2}{\sigma_T^2}\left[\mathrm{erf}(\gamma_{CMS}) + \frac{2}{\pi} I(a,b)\right] \exp\left[\alpha_{CMS}^2 - x_{CMS}^2\right] \quad \text{(B7)}$$

where

$$I(a,b) \equiv \frac{1}{a}\int_{-\infty}^{\infty} ds \exp\left[-\left(\frac{s}{a}\right)^2\right] \int_0^S dt \exp\left[-(t+b)^2\right] \quad \text{(B8)}$$

with $a \equiv \sigma_S/\sigma_H$, $b \equiv \gamma_{CMS}$, comes from the integration on du_z.

The $I(a,b)$ integral can be evaluated as follows: first, we derive it with respect to b, the we carry out the integration with respect to t, obtaining:

$$\frac{\partial I(a,b)}{\partial b} = \sqrt{\pi}\left[\frac{1}{(1+a^2)^{1/2}} \exp\left(-\frac{b^2}{1+a^2}\right) - \exp(-b^2)\right] \quad \text{(B9)}$$

and, finally, an integration on b leads to:

$$I(a,b) = I(a,0) + \frac{\pi}{2}\left[\mathrm{erf}\left[\frac{b}{(1+a^2)^{1/2}}\right] - \mathrm{erf}(b)\right] \quad \text{(B10)}$$

with $I(a,0) = 0$ because $\mathrm{erf}(s) = -\mathrm{erf}(-s)$. Substituting the expression for $I(a,b)$ in eq. (B7), we get:

$$I = \frac{\sigma_B^2}{\sigma_T^2} \exp\left(\alpha_{CMS}^2 - x_{CMS}^2\right) \mathrm{erf}(\alpha_{CMS}) \quad \text{(B11)}$$

where now $\boldsymbol{x}_{CMS} \equiv \boldsymbol{v}_{CMS}/\sqrt{2}\sigma_T$ and $\alpha_{CMS} \equiv \boldsymbol{x}_{CMS} \cdot \boldsymbol{\zeta}$.